MATH 307

Hw 4 posted

EXAM 1 REFLECTION DUE TODAY

3.3, 3.4 REVIEWS ARE POSTED AS WELL AS 3.1, 3.3, 3.4 SUMMARY

3.4: Repeated Roots / Reduction of Order

Recall if $y=e^{rt}$ is a sol'n to $ay''+by'+cy=0$
then $ar^2+br+c=0$

$b^2-4ac>0 \Rightarrow y(t)=c_1e^{r_1t}+c_2e^{r_2t}$

$b^2-4ac<0 \Rightarrow y(t)=a_1e^{(\lambda+\omega i)t}+a_2e^{(\lambda-\omega i)t}$
$= e^{\lambda t}(c_1\cos(\omega t)+c_2\sin(\omega t))$

$b^2-4ac=0 \Rightarrow y(t)=c_1e^{rt}+\underbrace{c_2te^{rt}}_{=???}$ TODAY!

Ex Consider $y''-6y'+9y=0$

$r^2-6r+9=0$

$(r-3)(r-3)=0$

$\boxed{r=3}$ is a repeated root

Thus, $y_1(t)=e^{3t}$ is one sol'n.

We need another independent sol'n

$y(t)=c_1e^{3t}+c_2\,???$

Method of Reduction of Order

WE WILL SEE THAT THE OTHER SOL'N HAS THE FORM $u(t)=???$

STEP 1 Guess $y_2(t)=u(t)y_1(t)=u(t)e^{3t}$.

STEP 2 Then $y_2'(t)=u'e^{3t}+3ue^{3t}=(u'+3u)e^{3t}$

$y_2''(t)=(u''+3u')e^{3t}+3(u'+3u)e^{3t}$
$=(u''+6u'+9u)e^{3t}$

STEP 3 Substitute

$$y'' - 6y' + 9y \stackrel{?}{=} 0$$

$$(u'' + 6u' + 9u)e^{3t} - 6(u' + 3u)e^{3t} + 9ue^{3t} \stackrel{?}{=} 0$$

$$u'' + \cancel{6u'} + \cancel{9u} - \cancel{6u'} - \cancel{18u} + \cancel{9u} \stackrel{?}{=} 0$$

$$u'' = 0 \quad \leftarrow \text{ order has been reduced}$$

$\boxed{\text{STEP 4}}$ $v = u' \Rightarrow u'' = v' = 0$

$$u' = v = a_1$$

$$u(t) = a_1 t + a_2$$

$\boxed{\text{STEP 5}}$ $y(t) = (a_1 t + a_2)e^{3t} = a_1 t e^{3t} + a_2 e^{3t}$

is a sol'n for any a_1, a_2.

Thus, $\boxed{y_2(t) = te^{3t}}$ is a second sol'n

$$W = \begin{vmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & e^{3t} + 3te^{3t} \end{vmatrix}$$

$$= e^{6t}(1+3t) - 3te^{6t} = e^{6t} \neq 0$$

$$\boxed{y(t) = c_1 e^{3t} + c_2 te^{3t}}$$

For repeated roots, $\boxed{y_1(t) = e^{rt} \text{ and } y_2(t) = te^{rt}}$

You do Solve $y'' + 20y' + 100y = 0$ $\quad y(0) = 3,\ y'(0) = 6$

$$r^2 + 20r + 100 \overset{?}{=} 0$$

$$(r+10)^2 = 0 \qquad \boxed{r = -10}$$

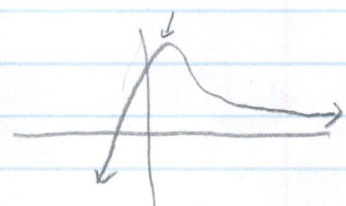
$$y(t) = c_1 e^{-10t} + c_2 t e^{-10t}$$ general sol'n.

$$y'(t) = -10c_1 e^{-10t} + c_2\left(e^{-10t} - 10te^{-10t}\right)$$

$$y(0) = 3 \Rightarrow c_1 + 0 = 3 \Rightarrow \boxed{c_1 = 3}$$

$$y'(0) = 6 \Rightarrow -10c_1 + c_2 = 6 \Rightarrow -30 + c_2 = 6 \qquad \boxed{c_2 = 36}$$

$$\boxed{y(t) = 3e^{-10t} + 36te^{-10t}}$$



$c_2 > 0$

Reduction of Order

CAN BE USED TO SOLVE GENERAL LINEAR EQUATIONS. 3.3/35,36 3.4/40

Ex CONSIDER $\boxed{2t^2y'' + 3ty' - y = 0}$ $\leftarrow$ Euler Equation

$t > 0$

See 3.3 HW for one method to solve THIS

BY GUESS AND CHECK, ONE SOL'N IS $\boxed{y_1(t) = \frac{1}{t}}$

$$y_1(t) = t^{-1}, \quad y_1'(t) = -t^{-2}, \quad y_1''(t) = 2t^{-3}$$

$$2t^2\,\frac{2}{t^3} + 3t\,\frac{-1}{t^2} - \frac{1}{t} = \frac{4-3-1}{t} = 0 \checkmark$$

FIND ANOTHER SOL'N!
(3.4/23, 24, 25)

$\boxed{\text{STEP 1}}$ $y = u(t)\, t^{-1} = u t^{-1}$

$\boxed{\text{STEP 2}}$ $y' = u' t^{-1} - u t^{-2}$, $y'' = u'' t^{-1} - u' t^{-2} - u' t^{-2} + 2 u t^{-3}$

$= u'' t^{-1} - 2u' t^{-2} + 2ut^{-3}$

$\boxed{\text{STEP 3}}$ SUBSTITUTE

$$2t^2 y'' + 3ty' - y \overset{?}{=} 0$$

$$2t^2\left(u'' t^{-1} - 2u' t^{-2} + 2ut^{-3}\right) + 3t\left(u' t^{-1} - u t^{-2}\right) - \left(u t^{-1}\right) = 0$$

$$2t\, u'' + (-4 + 3) u' + \underbrace{\left(4t^{-1} - 3t^{-1} - t^{-1}\right)}_{=0} u \overset{?}{=} 0$$

↑ ALWAYS WILL HAPPEN BECAUSE t^{-1} IS A SOL'N.

$$2t\, u'' - u' = 0$$

REDUCED THE ORDER!!!

$\boxed{\text{STEP 4}}$ $v = u' \Rightarrow$ $2tv' - v = 0$

$$2t \frac{dv}{dt} = v$$

$$\frac{1}{v}\frac{dv}{dt} = \frac{1}{2t}$$

$$\int \frac{1}{v}\, dv = \int \frac{1}{2t}\, dt$$

$$\ln|v| = \tfrac{1}{2}\ln|t| + C$$

$$v = \pm e^{C} e^{\frac{1}{2}\ln|t|}$$

$D = \pm e^{C}$

$$u'(t) = v(t) = D\, t^{1/2}$$

$D_1 = \frac{2}{3} D$

$$u(t) = D_1 t^{3/2} + D_2$$

$$y = u(t)\, t^{-1} = \left(D_1 t^{3/2} + D_2\right) t^{-1} = D_1 t^{1/2} + D_2 t^{-1}$$

$\boxed{\text{NEW!}}$ ↑   ↑ $y_1(t)$

$$\boxed{y_2(t) = t^{1/2}}$$

$$\boxed{y(t) = c_1 t^{-1} + c_2 t^{1/2}}$$